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Two-dimensional representations of the free group in two generators over an arbitrary field

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Abstract

Working over an arbitrary field, we classify all two-dimensional representations of the free group with two generators. © 2000 Published by Elsevier Science Inc. All rights reserved.

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0. Introduction

In their book [5, pp. 79–84], the authors give a classification of two-dimensional complex representations of the free group with two generators. We extend this classification to representations over an arbitrary field. A more general problem is the classification of pairs of matrices up to simultaneous conjugation. It was considered over the complex numbers by Friedland [4], and by other authors. Note that in [4], as well as in invariant theory in general, an invariant means a polynomial invariant and that the equality of all such invariants does not guarantee the similarity. All polynomial invariants for our problem and even for more general problems are known [1–4]. An interesting and very difficult question outside the scope of this paper (as well as of [5]) is how to recognize faithful representations among our representations.

Let $G = \langle u_1, u_2 \rangle$ be the free group generated by u_1, u_2; V a two-dimensional vector space over an arbitrary field F; and $\rho : G \to GL(V)$, a two-dimensional representation.

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Let $g_i = \rho(u_i)$ for $i = 1, 2$ and $g_3 = \rho(u_1u_2)^{-1}$. We set $t_i = \mathrm{tr}(g_i)$ and $e_i = \det(g_i)$ for $i = 1, 2, 3$. Since $g_1g_2g_3 = 1$, we have $e_1e_2e_3 = 1$.

If F is algebraically closed (or, more generally, quadratically closed), we will see (Theorem 1 below) that there are no other relations. In general, we describe all possible 5-tuples $(t_1, t_2, t_3, e_1, e_2)$ in F.

Theorem 1. *A two-dimensional representation ρ of G with given $(t_1, t_2, t_3, e_1, e_2)$ over F, where $e_1e_2 \neq 0$, exists if and only if the quadratic form*

$$x_1^2/e_1 + x_2^2/e_2 + x_3^2/e_3 + t_1x_2x_3 + t_2x_1x_3 + t_3x_1x_2$$

in three variables x_1, x_2, x_3 is isotropic.

Note that this theorem is similar to Proposition 9.1 in [6].

We now want to address the question of uniqueness of representations with given $(t_1, t_2, t_3, e_1, e_2)$. We will see that uniqueness of a representation is equivalent to its irreducibility.

Theorem 2. *Given ρ with traces and determinants $(t_1, t_2, t_3, e_1, e_2)$ in F, the following three conditions are equivalent*:

(a) *the representation ρ is reducible*;
(b) *the eigenvalues of all g_i $(i = 1, 2, 3)$ are in F and there are eigenvalues λ_i of the matrices g_i whose product is* 1;
(c) *ρ is not unique* (*up to similarity*).

Condition (b) could be stated in terms of the traces and determinants $(t_1, t_2, t_3, e_1, e_2)$ as well. (Namely, consider the monic polynomial f of degree 8 whose zeros are the eight products of the eigenvalues of g_i. Its coefficients are polynomials in t_i, e_i with integral coefficients. For example, the degree 7 coefficient is $-t_1t_2t_3$ and the constant term is 1. Then the product condition in (b) means that 1 is a zero of f, i.e., the sum of coefficients of f is 1.) This condition obviously holds when ρ is reducible, i.e., there is a ρ-invariant subspace $W = \rho(G)W \neq 0, V$ of V. What is not so obvious is that this is a sufficient condition for ρ to be reducible. When ρ is reducible, Theorem 2 says that there are at least two representations with the same $(t_1, t_2, t_3, e_1, e_2)$. For example, the reducible representation is either decomposable, i.e., V is the direct sum of invariant subspaces W and S; or the representation is indecomposable. In Section 3, we classify all reducible representations with given $(t_1, t_2, t_3, e_1, e_2)$.

In the following sections, we choose a basis for V so that the endomorphisms g_i of V become 2×2 matrices.

1. Proof of Theorem 1

Assume first that the quadratic form is isotropic, i.e., there are $x_i \in F$, not all of them zero, such that

$$x_1^2 + x_2^2 + x_3^2 + t_3x_1x_2 + t_2x_1x_3 + t_1x_2x_3 = 0.$$

If two of the x_i vanish, then all three equal zero. So we can assume that at most one of the x_i vanishes, say, $x_1x_2 \neq 0$. (The cases $x_1x_3 \neq 0$ and $x_2x_3 \neq 0$ can be reduced to this case by observing that $g_1(g_2g_3g_2^{-1})g_2 = 1_2$ and then observing that $g_2(g_2^{-1}g_1g_2)g_3 = 1_2$.)

Set

$$g_1 = \begin{pmatrix} t_1 + y & z \\ x_1 & -y \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & -e_2/x_2 \\ x_2 & t_2 \end{pmatrix}$$

with $z = (-e_1 - y(t_1 + y))/x_1$, where $y = x_3/x_2e_3$. Then $\mathrm{tr}(g_i) = t_i$ and $\det(g_i) = e_i$ for $i = 1, 2$. Note that the matrices g_1 and g_2 are not scalars because $x_1x_2 \neq 0$.

We set $g_3 = (g_1g_2)^{-1}$. This matrix has the correct determinant $e_3 = (e_1e_2)^{-1}$ automatically. We have to check that it has the correct trace t_3. This is equivalent to $(g_3)^{-1} = g_1g_2$ having trace t_3/e_3. The straightforward computation shows that the equality $\mathrm{tr}(g_1g_2) = zx_2 - e_2x_1/x_2 - t_2y = t_3/e_3$ becomes $x_1^2 + x_2^2 + x_3^2 + t_3x_1x_2 + t_2x_1x_3 + t_1x_2x_3 = 0$ after multiplication by $x_1x_2e_3$.

Assume now that $g_1g_2g_3 = 1_2$. We will show that the quadratic form is isotropic. If not all g_i are scalars, then, without loss of generality, we can assume that g_2 is not scalar. Choosing an appropriate basis of V, we can assume that $(g_2)_{1,1} = 0$. Set $x_i = (g_i)_{2,1} \in F$. If $x_1 = 0$, then $x_3 \neq 0$ and replacing g_1, g_2, g_3 by $g_3, g_2, g_2^{-1}g_1g_2$, we can assume that $x_1 \neq 0$. We write

$$g_1 = \begin{pmatrix} t_1 + y & z \\ x_1 & -y \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & -e_2/x_2 \\ x_2 & t_2 \end{pmatrix}.$$

Since $-yx_2 = (g_1g_2)_{2,1} = (g_3^{-1})_{2,1} = -x_3/e_3$, it follows that $y = x_3/(x_2e_3)$. Writing $\mathrm{tr}(g_1g_2) = \mathrm{tr}(g_3^{-1}) = t_3/e_3$, we obtain

$$x_1^2 + x_2^2 + x_3^2 + t_3x_1x_2 + t_2x_1x_3 + t_1x_2x_3 = 0.$$

To finish the proof, it remains to show that the quadratic form is isotropic when all g_i are scalars. In this case, $g_i = \lambda_i 1_2$, $e_i = \lambda_i^2$, $t_i = 2\lambda_i$, and the quadratic form becomes $(x_1/\lambda_1 + x_2/\lambda_2 + x_3/\lambda_3)^2$, which is isotropic. $\square$

2. Proof of Theorem 2

If ρ is reducible, all matrices are upper triangular in a certain basis of V. Thus, $\lambda_1\lambda_2\lambda_3 = 1$ for the first eigenvalues λ_i of g_i (the product of the second eigenvalues is also 1), and all eigenvalues are in F. So it is obvious that (a) implies (b).

Assume now (b). We want to prove (a). The characteristic polynomial of g_i is $x^2 - t_ix + e_i$. Let λ_i be as in (b), so $\lambda_1\lambda_2\lambda_3 = 1$.

We set

$$g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So

$$\lambda_2^2 - (a+d)\lambda_2 + e_2 = 0. \tag{2.1}$$

If $bc = 0$, then the representation is reducible and we are done. Suppose now that $bc \neq 0$. Since the eigenvalues of g_1 are in F, we can assume that either

$$g_1 \text{ is a nonscalar diagonal matrix, } \operatorname{diag}(\lambda_1, \mu_1), \tag{2.2}$$

or

$$g_1 \text{ is a nonscalar upper triangular matrix with equal eigenvalues,} \tag{2.3}$$

or

$$g_1 \text{ is a scalar matrix.} \tag{2.4}$$

In case (2.2), since $\lambda_1\lambda_2$ is an eigenvalue for $g_1 g_2 = g_3^{-1}$, it annihilates the characteristic polynomial:

$$(\lambda_1\lambda_2)^2 - (\lambda_1 a + \mu_1 d)(\lambda_1\lambda_2) + e_2\lambda_1\mu_1 = 0. \tag{2.5}$$

Multiplying (2.1) by λ_1^2 and subtracting (2.5) from the result gives

$$\lambda_1(\mu_1 - \lambda_1)(d\lambda_2 - e_2) = 0.$$

From $\mu_1 \neq \lambda_1$ and $e_2 = \lambda_2 d = \lambda_2\mu_2$, we obtain that $\mu_2 = d$. Thus, $a = \lambda_2$ and we conclude that $bc = 0$.

In case (2.3),

$$g_1 = \begin{pmatrix} \lambda_1 & z \\ 0 & \lambda_1 \end{pmatrix}$$

with $z \neq 0$.

The trace of $g_1 g_2$ is then equal to $\lambda_1(a+d) + zc$ where $zc \neq 0$.

So the eigenvalue $\lambda_1\lambda_2$ of $g_1 g_2 = g_3^{-1}$ is a root of two quadratic polynomials: one is the characteristic polynomial for $g_1 g_2$ involving the above trace, and the other one is the characteristic polynomial for $\lambda_1 g_2$ whose trace is $\lambda_1(a+d)$. The constant coefficients of both monic polynomials are $e_1 e_2$. So $\lambda_1\lambda_2 = 0$, but this is impossible for an eigenvalue of an invertible matrix.

In case (2.4), it is clear that g_1, g_2 are simultaneously similar to upper triangular matrices, and we are done, because then ρ is reducible.

So we have proved that (a) and (b) are equivalent. Assume (a), and let us prove (c). Since ρ is reducible we can assume that all g_i are upper triangular. Replacing the off-diagonal entries of g_1 and g_2 by zeros and ones, we obtain four different (not similar) representations of G with the same $(t_1, t_2, t_3, e_1, e_2)$.

We now have to prove that (c) implies (a) or, equivalently, that an irreducible ρ is similar to any representation ρ' with the same $(t_1, t_2, t_3, e_1, e_2)$. (Unlike [5], we do not assume that ρ' is irreducible.)

If ρ' is scalar, then (b) holds. Hence (a) holds for any representation with given $(t_1, t_2, t_3, e_1, e_2)$. This contradicts the assumption that $(t_1, t_2, t_3, e_1, e_2)$ came from an irreducible representation.

So we can assume that ρ' is not scalar, i.e., either g'_1 or g'_2 is not scalar, where $g'_i = \rho'(u_i)$. These two cases are similar, and so we consider only the case when g'_2 is not scalar.

If one of the other g'_i is scalar, then we have the product condition from (b). If the corresponding g_i is not scalar, then g_i is similar to an upper triangular nonscalar matrix with equal eigenvalues and, as was seen in the proof of case (2.3), we get a contradiction (without using that the eigenvalues are in F).

So g_i is similar to g'_i for $i = 1, 2, 3$. We now conclude by rigidity [7] that ρ and ρ' are similar. $\square$

3. Reducible case

In this section, we assume that ρ is reducible. In terms of eigenvalues, this means (see Theorem 2) that the eigenvalues of g_i are in the ground field F and there are eigenvalues λ_i of g_i such that $\lambda_1\lambda_2\lambda_3 = 1$. Let μ_i be the other eigenvalue of g_i for $i = 1, 2, 3$. Then $\mu_1\mu_2\mu_3 = 1$ too. We want to classify ρ (up to similarity) with given $\{\lambda_i, \mu_i\} \subset F$ for $i = 1, 2, 3$. The classification and proofs are the same as in the case when F is the field of complex numbers (see [5]). So we give the classification without proofs.

First of all, given $\lambda_i, \mu_i \in F$ as above, ρ could be decomposable. In other words, all matrices of ρ would be diagonal in some basis. It is clear that a decomposable representation ρ with given $\lambda_i, \mu_i \in F$ is unique up to similarity.

Now we describe all indecomposable reducible ρ.

Case 3.1: $\lambda_1 \neq \mu_1$, i.e., $t_1^2 \neq 4e_1$, so g_1 is not scalar.

Then there are two conjugacy classes of ρ given by

- $g_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} \lambda_2 & 1 \\ 0 & \mu_2 \end{pmatrix}$,
- $g_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} \lambda_2 & 0 \\ 1 & \mu_2 \end{pmatrix}$.

Case 3.2: $\lambda_1 = \mu_1$ and $\lambda_2 \neq \mu_2$.

Then there are two conjugacy classes of ρ given by

- $g_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \mu_2 \end{pmatrix}$,
- $g_1 = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \mu_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \mu_2 \end{pmatrix}$.

Note that Cases 3.1 and 3.2 are impossible if and only if card(F) = 2.

Case 3.3: Each of the three matrices g_i has equal eigenvalues, i.e., $t_i^2 = 4e_i$ for $i = 1, 2, 3$.

Then the representations ρ are classified by the projective line over F. Namely, there is a one-parameter family

- $g_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} \lambda_2 & b \\ 0 & \mu_2 \end{pmatrix}$ $(b \in F)$,

  as well as the representation
- $g_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} \lambda_2 & 1 \\ 0 & \mu_2 \end{pmatrix}$.

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